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A new interpretation of the orthofermion algebra

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Abstract
We present particle algebras whose representations correspond to states having at most p particles. For $p = 1$, the algebra corresponds to fermions. For $p = 2, 3, 4, \ldots$, the algebra corresponds to the orthofermion algebra $\mathcal{C}_p$ with a new interpretation.

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1. Introduction

Oscillator algebras and more recently their deformations have received a lot of attention. The fermion algebra

$$aa^* + a^*a = 1, \qquad a^2 = 0 \tag{1}$$

has a unique two-dimensional representation which physically corresponds to the exclusion principle obeyed by fermions, whereas the boson algebra

$$aa^* - a^*a = 1 \tag{2}$$

has the unique infinite-dimensional representation. In both cases the number operator $N = a^*a$ counts the possible number of particles in a given state. Spec $N = \{0, 1\}$ for fermions whereas Spec $N = \{0, 1, 2, \ldots\}$ for bosons. An immediate question which arises is whether there exists a particle algebra where the number operator N has the spectrum $\{0, 1, \ldots, p\}$, $p \in \mathcal{Z}^+$. Nondeformed [1] and deformed [2] parafermions of order p satisfy this criterion. It has also been recognized that the answer to this question is partially given [3] by the orthofermion algebra [4–7] $\mathcal{C}_p$ which is generated by annihilation and creation operators c_i, c_i^* which satisfy the following relations:

$$c_i c_j^* + \delta_{ij} \sum_{k=1}^{p} c_k^* c_k = \delta_{ij}, \tag{3}$$

$$c_i c_j = 0, \qquad c_i^* c_j^* = 0, \qquad i, j \in \{1, 2, \ldots, p\}, \tag{4}$$

where δ_{ij} stands for the Kronecker delta function. The usual number operator of the orthofermion algebra is defined by $N = c_1^* c_1 + c_2^* c_2 + \cdots + c_p^* c_p$ so that Spec $N = \{0, 1\}$. However, if the number operator is defined by

$$N = c_1^* c_1 + 2c_2^* c_2 + \cdots + p c_p^* c_p, \tag{5}$$

then N has the desired spectrum $\{0, 1, \ldots, p\}$. We will show that creation and annihilation operators a^*, a can be defined such that the number operator in (5) can be written as $N = a^* a$. Orthofermion algebras are important because they can be used for constructing local relativistic quantum field theory [6]. Another important aspect of orthofermions is that although they are not deformed they are invariant under the quantum group $SU_q(p+1)$ for any deformation parameter q [8]. Furthermore, orthofermions are related to topological symmetries [9]. A matrix representation of orthofermions of order p is given by $(p+1) \times (p+1)$ matrices with entries

$$[c_i]_{kl} = \delta_{k1} \delta_{li+1}, \qquad i, k, l \in \{1, 2, \ldots, p+1\}. \tag{6}$$

This is a unique nontrivial $(p+1)$-dimensional irreducible representation [9]. Orthofermions of order 1 are usual fermions as in (1).

On the other hand, comparing the mathematical relations in (3), (4) with (1) and (2) one immediately recognizes that these look very different. To interpret the orthofermion algebra $\mathcal{C}_p$ as a single-particle algebra one has to define a single-creation operator a^* and a single-annihilation operator a. In this paper, we will present the algebra $\mathcal{A}_p$ which is generated by the annihilation operator a and creation operator a^* which satisfy the following relations:

$$aa^* - a^* a = 1 - \frac{p+1}{p!} a^{*p} a^p, \qquad a^{p+1} = 0. \tag{7}$$

By defining

$$a = c_1 + \sqrt{2} c_1^* c_2 + \cdots + \sqrt{p} c_{p-1}^* c_p, \tag{8}$$

it is indeed straightforward to show that (3)–(4) imply (7). What is nontrivial, however, is that by using the a and a^* of $\mathcal{A}_p$ defined by (7), all the c_i and c_i^* of (3) and (4) can be defined and satisfy these relations. We will accomplish this and will show that this algebra is isomorphic to the orthofermion algebra. Furthermore, in the representation of orthofermion algebra the states $c_n^*|0\rangle$ correspond to a state containing the nth orthofermion, whereas in the representation of $\mathcal{A}_p$ the corresponding state will be the n-particle state, i.e. a state containing n particles. The creation and annihilation operators of all the algebras $\mathcal{A}_p$ defined by (7), upon multiplication of the first relation by a from the right, satisfy

$$aa^* a - a^* aa = a. \tag{9}$$

One can take this relation to define an algebra $\mathcal{A}$. This algebra is interesting since it possesses both finite-dimensional and infinite-dimensional representations. Finite-dimensional representations correspond to orthofermions and the infinite-dimensional representation corresponds to bosons.

This paper is organized as follows. In section 2, we will prove that there exists an isomorphism from $\mathcal{A}_p$ onto the orthofermion algebra $\mathcal{C}_p$ for all p. In section 3, we will show that there exists an infinite-dimensional irreducible representation of the algebra defined in (9) as well as finite-dimensional irreducible representations of any dimension. Section 4 is reserved for our conclusions.

2. Construction of the algebra $\mathcal{A}_p$

Let us consider an algebra $\mathcal{A}_p$, generated by a, a^*, for each $p \in \mathcal{N}$ satisfying

$$a^{p+1} = 0, \qquad aa^* - a^*a = 1 - \beta a^{*p} a^p, \tag{10}$$

where β is a real number which may depend on p. Let $M = aa^*$ and $N = a^*a$. The algebra implies the following:

$$aa^*a - a^*aa = a, \tag{11}$$

$$aN = (N+1)a, \tag{12}$$

$$a^{*k}a^k = N(N-1)(N-2)\cdots(N-k+1), \qquad k = 1, 2, \ldots, p \tag{13}$$

$$(N-p)a^{*p}a^p = 0, \tag{14}$$

$$(a^{*p}a^p)^2 = p!a^{*p}a^p. \tag{15}$$

Multiplying equation (11) from the right by a^* and taking conjugate of the resulting equation gives us two equations

$$M^2 - NM = M, \tag{16}$$

$$M^2 - MN = M. \tag{17}$$

These imply that M and N commute with each other. Then we will use equation (16) in the following lemma to obtain values of β.

Lemma 1. *If, for each $p \in \mathcal{N}$, an algebra generated by a, a^* satisfies $a^{p+1} = 0$ and $aa^* - a^*a = 1 - \beta a^{*p}a^p$ then $\beta = \frac{p+1}{p!}$.*

Proof.

$$M = (N+1) - \beta a^{*p}a^p. \tag{18}$$

Equation (16) gives

$$\begin{aligned} M^2 &= (N+1)^2 - 2\beta(N+1)a^{*p}a^p + \beta^2(a^{*p}a^p)^2 \\ &= (N+1) - \beta a^{*p}a^p + N(N+1) - \beta N a^{*p}a^p \end{aligned} \tag{19}$$

and then equations (14), (15) simplify (19) as

$$[-2\beta(p+1) + \beta^2 p! + \beta(1+p)]a^{*p}a^p = 0 \tag{20}$$

$$\beta[-(1+p) + p!\beta] = 0. \tag{21}$$

Hence $\beta = 0$ or $\beta = \frac{p+1}{p!}$. The case $\beta = 0$ with the assumption $a^{p+1} = 0$ contradicts the relation $aa^* - a^*a = 1$. Therefore the only β is $\frac{p+1}{p!}$.

For $p = 1$, the algebra $\mathcal{A}_p$ is the usual fermion algebra as in (1). For $p \geqslant 1$ we define operators

$$\Pi_k = \frac{(-1)^{p-k}}{k!(p-k)!} \prod_{\substack{j=0 \\ j \neq k}}^{p} (N-j), \qquad k = 0, 1, \ldots, p. \tag{22}$$

We will show that these are projection operators. □

Lemma 2. *$\Pi_k \in \mathcal{A}_p$, defined in (22) are projection operators and satisfy the following identities:*

1. $\Pi_0 + \Pi_1 + \cdots + \Pi_p = 1$
2. $N = \sum_{k=0}^{p} k\Pi_k$
3. $M = \sum_{k=1}^{p} k\Pi_{k-1}$
4. $(N-k)\Pi_k = \Pi_k(N-k) = 0 \quad k = 0, 1, 2, \ldots, p$
5. $\Pi_k\Pi_l = \delta_{kl}\Pi_k \quad k, l = 1, 2, \ldots, p$
6. $M\Pi_k = \Pi_k M = (k+1)\Pi_k$
7. $a\Pi_k = \Pi_{k-1}a \quad k = 1, 2, \ldots, p.$

Proof. Identities 1 and 2 can be obtained just by substitution from definitions (22). For identity 3, comparing (13) for $k = p$ with (22) gives us

$$p!\Pi_p = a^{*p}a^p. \tag{23}$$

Substituting this, and identities 2 and 1 in

$$M = (N+1) - \frac{p+1}{p!}a^{*p}a^p \tag{24}$$

provides us identity 3. For identity 4, equations (14), (23) imply

$$N\Pi_0 = 0 \tag{25}$$

for $k = 0$. For the rest of k use (12) to obtain a factor $\Pi_p a$ which is zero. Then the result is obtained. The commutativity in identity 4 is obvious. Identities 4 and 1 prove identity 5. Identities 3 and 5 imply identity 6. For identity 7, we use equation (12), identities 1, 2 and

$$\Pi_p a = 0, \qquad a\Pi_0 = 0. \tag{26}$$

Then

$$a(\Pi_1 + 2\Pi_2 + \cdots + p\Pi_p) = (\Pi_0 + 2\Pi_1 + \cdots + p\Pi_{p-1})a \tag{27}$$

$$\Pi_k a \Pi_l = 0, \qquad l - k \neq 1, \qquad k, l = 0, 1, \ldots, p \tag{28}$$

$$a\Pi_k = \Pi_{k-1}a\Pi_k = \Pi_{k-1}a. \tag{29}$$

□

Theorem. *The algebra $\mathcal{A}_p$ is isomorphic to the orthofermion algebra C_p.*

Proof. Let us define a map $\rho : \mathcal{A}_p \to C_p$ which is linear, *-preserving, multiplicative, $\rho(1) = 1$, determined by mapping generators to generators

$$\rho(a_k) = c_k, \qquad k = 1, 2, \ldots, p, \tag{30}$$

where

$$a_k = \frac{1}{\sqrt{k!}}\Pi_0 a^k. \tag{31}$$

This shows that ρ is an algebra isomorphism, i.e. ρ is a one to one and onto mapping. For this we will prove that the generators of $\mathcal{A}_p$ satisfy the relations for those of C_p. Using (31) and identities 7, 6, 5, 4 in lemma 2 several times we obtain

$$a_k^* a_k = \Pi_k, \qquad a_k a_k^* = \Pi_0, \qquad k = 1, 2, \ldots, p, \tag{32}$$

$$a_i a_j^* = 0 \quad i \neq j, \qquad a_i a_j = 0, \qquad a_i^* a_j^* = 0, \qquad i, j = 1, 2, \ldots, p. \tag{33}$$

Now we can explicitly give the basis which makes this isomorphism possible

$$\{x \in \mathcal{A}_p | x = \Pi_k, \Pi_k a^j, a^{*j}\Pi_k, k = 0, 1, \ldots, p, j = 1, 2, \ldots, p-k\}. \tag{34}$$

This amounts to $(p+1)^2$ linearly independent elements and hence it is a basis for $\mathcal{A}_p$. Consider a set which contains all monomials in $\{a_k, a_k^*\}_{k=1}^p$, in which monomials are normal ordered, as generators of $\mathcal{A}_p$. This will be the same set as in (34) except the coefficients. Hence the algebra $\mathcal{A}_p$ with generators $\{a_k, a_k^*\}_{k=1}^p$ is the orthofermion algebra. Having obtained an isomorphism from $\mathcal{A}_p$ to C_p we can search for the opposite, i.e. 'how can the generators of C_p be written to obtain the algebra $\mathcal{A}_p$?'. To do this we consider an element in C_p such that

$$f = c_1 + \sum_{k=2}^{p} \sqrt{k} c_{k-1}^* c_k. \tag{35}$$

This satisfies

$$f^p = \sqrt{p!} c_p, \qquad c_p^* c_p = \frac{f^{*p} f^p}{p!}. \tag{36}$$

Here, note that the integer p is the subindex of c and c^*, whereas it is the power of f and f^*. Hence we have obtained all that is needed to prove the relations

$$ff^* - f^* f = 1 - \frac{p+1}{p!} f^{*p} f^p, \qquad f^{p+1} = 0. \tag{37}$$

□

3. Representations of the algebra $aa^*a - a^*aa = a$

It is well known that the boson algebra (2) has a unique infinite-dimensional irreducible representation, which, in our notation corresponds to

$$a|n, n+1\rangle = \sqrt{n}|n-1, n\rangle \tag{38}$$

$$a^*|n, n+1\rangle = \sqrt{n+1}|n+1, n+2\rangle, \qquad n = 0, 1, 2, \ldots. \tag{39}$$

If the defining relation (2) for the boson algebra is multiplied by a from the right then we obtain the defining relation (9) for $\mathcal{A}$. This shows that $\mathcal{A}$ possesses an infinite-dimensional representation. We also obtain the algebra $\mathcal{A}$ just by multiplying the first equation of (7) by a from the right. This gives us existence of the finite-dimensional representations of $\mathcal{A}$ by means of the representations of orthofermions. These are in hand.

Now we will construct representations starting from simultaneous eigenvectors of $N = a^*a$ and $M = aa^*$, $|\nu, \mu\rangle$ since N and M are commutative

$$N|\nu, \mu\rangle = \nu|\nu, \mu\rangle \tag{40}$$

$$M|\nu, \mu\rangle = \mu|\nu, \mu\rangle. \tag{41}$$

For a non-negative integer k:

$$N(a^k|\nu, \mu\rangle) = (\nu - k)a^k|\nu, \mu\rangle. \tag{42}$$

Since N cannot have a negative eigenvalue there must exist a number n for which

$$\nu = n \tag{43}$$

and N has zero eigenvalue corresponding to the vector $|0, \mu\rangle$. We will show that in fact for $\nu = 0, \mu = 0$ or 1. To show this, let us determine eigenvalues of M corresponding to the vectors $|n, \mu\rangle$. Take equation (16) and apply it to the vector $|n, \mu\rangle$. Then

$$M^2|n, \mu\rangle - NM|n, \mu\rangle - M|n, \mu\rangle = 0 \tag{44}$$

$$\mu^2|n, \mu\rangle - n\mu|n, \mu\rangle - \mu|n, \mu\rangle = 0 \tag{45}$$

$$\mu(\mu - n - 1) = 0 \tag{46}$$

$$\mu = 0 \quad \text{or} \quad \mu = n + 1. \tag{47}$$

The customary way to build representations of an oscillator algebra is to start from the ground state $|0, 1\rangle$ which is annihilated by a and to build the other states in the representation by applying the creation operator a^*. However this method fails for $\mathcal{A}$ since there may exist states which are annihilated by a^*. More explicitly, it is impossible to deduce the M eigenvalue of the state $a^*|0, 1\rangle$ by using (9).

On the other hand, we can obtain a unique finite $(n + 1)$-dimensional irreducible representation for any non-negative integer n. To show this we start with the vector $|n, 0\rangle$. For $n = 0$, $a|0, 0\rangle = 0$ and $a^*|0, 0\rangle = 0$ hence we have the only vector $|0, 0\rangle$ and one-dimensional trivial representation. Now for any fixed $n > 0$ take the vector $|n, 0\rangle$. The algebra $\mathcal{A}$ leads us to obtain $n + 1$ linearly independent sets of vectors

$$S_n = \{|n, 0\rangle, |n - 1, n\rangle, |n - 2, n - 1\rangle, \ldots, |1, 2\rangle, |0, 1\rangle\} \tag{48}$$

which is invariant under the actions of generators a, a^*. The proof follows

$$a^*|n, 0\rangle = 0 \tag{49}$$

which immediately follows from the fact that the norm of this vector is zero since $|n, 0\rangle$ is an eigenvector of $M = aa^*$ with eigenvalue zero. Then we calculate eigenvalues of N and M on $a|n, 0\rangle$:

$$Na|n, 0\rangle = a(N - 1)|n, 0\rangle = (n - 1)a|n, 0\rangle \tag{50}$$

$$Ma|n, 0\rangle = na|n, 0\rangle. \tag{51}$$

Thus we can set

$$a|n, 0\rangle = \alpha|n - 1, n\rangle, \tag{52}$$

where α is a normalization factor. Taking the norm of both sides

$$n = |\alpha|^2 \tag{53}$$

so that we can choose

$$a|n, 0\rangle = \sqrt{n}|n - 1, n\rangle. \tag{54}$$

It then follows that

$$a^*|n - 1, n\rangle = \sqrt{n}|n, 0\rangle. \tag{55}$$

Then we calculate eigenvalues of N and M on $a|n - 1, n\rangle$. We use the similar procedure above to get

$$a|n - 1, n\rangle = \sqrt{n - 1}|n - 2, n - 1\rangle \tag{56}$$

and

$$a^*|n-2, n-1\rangle = \sqrt{n-1}|n-1, n\rangle. \tag{57}$$

Then going on this way we obtain that

$$a|k-1, k\rangle = \sqrt{k-1}|k-2, k-1\rangle, \qquad k = n, n-1, \ldots, 1 \tag{58}$$

and

$$a^*|k-2, k-1\rangle = \sqrt{k-1}|k-1, k\rangle. \tag{59}$$

Thus we have $n + 1$ different eigenvalues of N and M with $n + 1$ linearly independent eigenvectors. We also found that starting from the eigenvector $|n, 0\rangle$ and acting each time with the operator a on the resulting vector we end up with the eigenvector $|0, 1\rangle$ which is annihilated by a. One can show that a vector $|k-1, k\rangle$ obtained from $|m, 0\rangle$ by applying the annihilation operator a $m - k + 1$ times and the vector $|k-1, k\rangle$ obtained from $|n, 0\rangle$ $(n \neq m)$ by applying a $n - k + 1$ times $(k < n, m)$ are in fact orthogonal. So to differentiate these vectors we will use a subindex for the vectors to indicate the top vector $|m, 0\rangle$ to which they are related. Thus we denote the set of vectors related to $|m, 0\rangle$ by

$$S_m = \{|m, 0\rangle, |m-1, m\rangle_m, |m-2, m-1\rangle_m, \ldots, |1, 2\rangle_m, |0, 1\rangle_m\} \tag{60}$$

which are invariant under the action of a, a^*. For each $m, m < n$, S_n and S_m become orthogonal subspaces. So we obtain reducible representations since the sets do not interfere each other under the actions of a, a^*. Hence once we have chosen n we obtain an $(n + 1)$-dimensional irreducible representation.

It is important to realize that although starting from a vector $|n, 0\rangle$ leads us to an $(n + 1)$-dimensional representation, starting from a vector $|n-1, n\rangle$ leads us nowhere since action of a^* on this vector may lead to a vector which is not an eigenvector of M. Thus it is possible that representations of $\mathcal{A}$ other than the ones we have presented exist.

4. Conclusion

We wanted to find an algebra for which the number operator N would have finite spectrum different from $\{0, 1\}$ as in the fermion algebra. We found a new presentation $\mathcal{A}_p$

$$aa^* - a^*a = 1 - \frac{p+1}{p!}a^{*p}a^p, \qquad a^{p+1} = 0$$

for the orthofermion algebra $\mathcal{C}_p$ whose representations have states $c_n^*|0\rangle$ corresponding to a state containing the nth orthofermion, whereas in the representation of $\mathcal{A}_p$ the corresponding state is the n-particle state, i.e. a state containing n particles.

We investigated a particle algebra $aa^*a - a^*aa = a$ showing that it has both infinite- and finite-dimensional representations. We obtained this result by constructing $(n+1)$-dimensional irreducible representation for any fixed non-negative integer n. Hence we see that it includes the fermion representation and the orthofermion representations. On the other hand, the method we used for constructing representations does not give information about infinite-dimensional representations except that which we know by means of the representation of the boson algebra.

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